Announcements

1) HW 5 up on CTools, due next Tuesday

2) Advising session 3-4, CB 2047 Pizza

3) Another Job Candidate in tomorrow 3-4 CB 2090

Recall: A point x in a metric space X is a limit point of $S \subseteq X$ provided $\forall\ \varepsilon > 0$, $B(x, \varepsilon) \cap S$ contains points other than x.

S is closed if it contains all its limit points

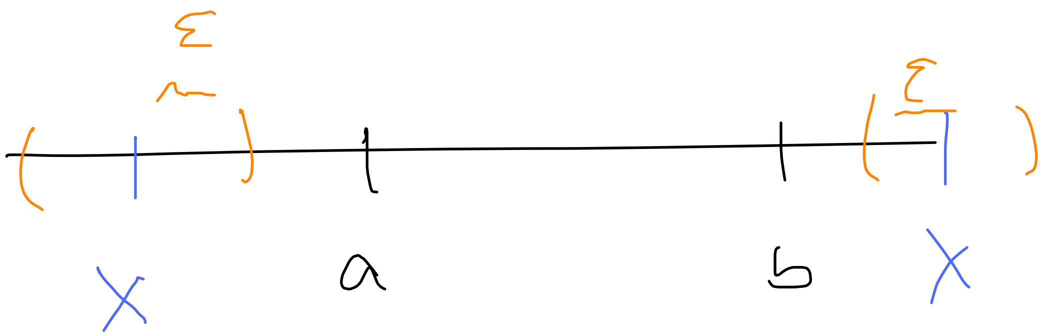
Example 1 ($\mathbb{R}$)

Any closed interval is a closed set – if $[a,b] \subseteq \mathbb{R}$, then if $x \notin [a,b]$, then either $x < a$ or $x > b$.

If $x < a$, with $\varepsilon = \dfrac{a-x}{2}$,

$B(x, \varepsilon) \cap [a,b] = \emptyset$.

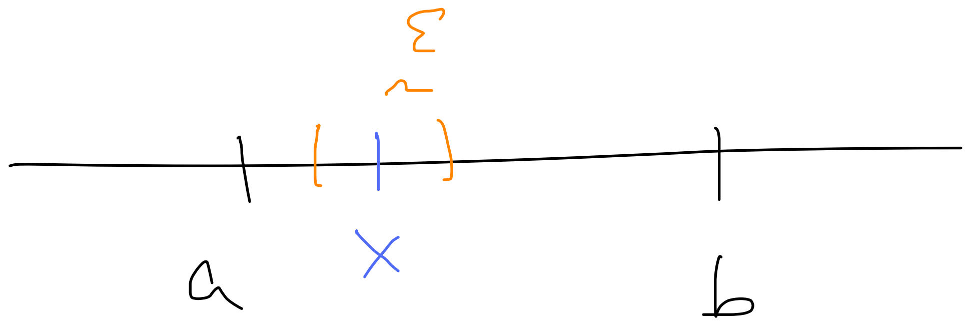
If $x > b$, with $\varepsilon = \dfrac{x-b}{2}$,



If $x > b$, with $\varepsilon = \frac{x-b}{2}$,

$$B(x, \varepsilon) \cap [a, b] = \emptyset$$

So if $x \notin [a, b]$, then x is not a limit point for $[a, b]$.



If $a < x < b$, let

$$\varepsilon = \frac{\min\{x-a,\, b-x\}}{2}.$$

Then $B(x, \varepsilon) \subseteq [a,b]$,

and in particular contains elements other than x.

Just like $(0,1)$,

$x=a$ and $x=b$ are limit points for (a,b), hence they are limit points for $[a,b]$. We have

then that $[a,b]$ is closed.

Example 2: Is $\mathbb{Q}$ either open or closed in $\mathbb{R}$?

If $x \in \mathbb{Q}$, is it true that $\exists$ $\varepsilon > 0$ (depends on x) with $B(x, \varepsilon) \subseteq \mathbb{Q}$?

By density of the irrationals, $\exists$ $y \in \mathbb{R} \setminus \mathbb{Q}$ with $|x - y| < \varepsilon$, so $\mathbb{Q}$ is not open.

However, $\mathbb{Q}$ is not closed since every $x \in \mathbb{R} \setminus \mathbb{Q}$ is a limit point! (careful proof later)

Proposition: (sequential limit point)

A point $x \in \overline{X}$ is a limit point of $S \subseteq \overline{X}$ if and only if $\exists\ (x_n)_{n=1}^{\infty} \subseteq S$, $x_n \neq x\ \forall\ n \in \mathbb{N}$ and

$$\boxed{\lim_{n \to \infty} x_n = x}$$

proof: $\Rightarrow$ Suppose x is a limit point of S.

Then with $\varepsilon_n = \frac{1}{n}$,

$\exists \ x_n \in S, \ x_n \neq x,$

$d(x, x_n) < \varepsilon_n = \frac{1}{n}.$

(definition of a limit point),

This gives us our sequence

$(x_n)_{n=1}^{\infty} \subseteq S$, and by

construction,

$\lim_{n \to \infty} x_n = x$.

$\Leftarrow$ Suppose $\exists\ (x_n)_{n\in\mathbb{N}} \subseteq S$,

$x_n \neq x$, $\lim\limits_{n\to\infty} x_n = x$.

Let $\varepsilon > 0$.

Since $\lim\limits_{n\to\infty} x_n = x$, $\exists\ N \in \mathbb{N}$

so that $d(x_n, x) < \varepsilon$ for

all $n \geq N$. This implies

$x_N \in B(x, \varepsilon)$ and $x_N \in S$,

$x_N \neq x$, so x is a limit

point of S. $\square$

Going back to $\mathbb{Q}$ example

Let $x \in \mathbb{R} \setminus \mathbb{Q}$.

By density of the rationals,

$\forall \varepsilon > 0, \exists y \in \mathbb{Q},$

$|x - y| < \varepsilon$

With $\varepsilon = \frac{1}{n}$, we produce

a sequence $(y_n)_{n \in \mathbb{N}} \subseteq \mathbb{Q}$,

$\lim_{n \to \infty} y_n = x$. Therefore,

x is a limit point of $\mathbb{Q}$,

(so $\mathbb{Q}$ is not closed)

Theorem: (closed/open)

A subset S of a metric space X is closed if and only if S^c is open.

proof: $\Rightarrow$ Suppose S is closed. Show S^c is open.

Let $x \in S^c$. Then x is not a limit point of S

This means $\exists$ $\varepsilon > 0$ with $B(x, \varepsilon) \cap S$ containing no points other than x. But $x \notin S$, so this says $B(x, \varepsilon) \cap S = \emptyset$, which shows S^c is open since then $B(x, \varepsilon) \cap S = \emptyset$ $\Rightarrow$ $B(x, \varepsilon) \subseteq S^c$.

$\Leftarrow$ Suppose S^c is open.

Let x be a limit point of S. So by the previous proposition, $\exists\ (x_n)_{n\in\mathbb{N}} \subseteq S$, $x_n \neq x$, $\lim_{n\to\infty} x_n = x$. Consider $\varepsilon > 0$ $B(x,\varepsilon)$. Then $\exists\ N \in \mathbb{N}$, $d(x_n, x) < \varepsilon\ \forall\ n \geq N$

So in particular,

$x_N \in B(x, \varepsilon)$.

But $x_N \in S \Rightarrow B(x,\varepsilon) \cap S \neq \emptyset$

$\Rightarrow B(x,\varepsilon) \nsubseteq S^c$.

This shows that there is no $\varepsilon > 0$ with $B(x,\varepsilon) \subseteq S^c$, hence $x \notin S^c$. Therefore, $x \in S$ and S is closed.

$\square$